Innovation Configuration for Mathematics

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Innovation Configuration for Mathematics

This paper features an innovation configuration (IC) matrix that can guide teacher preparation professionals in the development of appropriate content for mathematics. This matrix appears in the Appendix.

An IC is a tool that identifies and describes the major components of a practice or innovation. With the implementation of any innovation comes a continuum of configurations of implementation from non-use to the ideal. ICs are organized around two dimensions: essential components and degree of implementation (Hall & Hord, 1987; Roy & Hord, 2004). Essential components of the IC—along with descriptors and examples to guide application of the criteria to course work, standards, and classroom practices—are listed in the rows of the far left column of the matrix. Several levels of implementation are defined in the top row of the matrix. For example, no mention of the essential component is the lowest level of implementation and would receive a score of zero. Increasing levels of implementation receive progressively higher scores.

ICs have been used in the development and implementation of educational innovations for at least 30 years (Hall & Hord, 2001; Hall, Loucks, Rutherford, & Newton, 1975; Hord, Rutherford, Huling-Austin, & Hall, 1987; Roy & Hord, 2004). Experts studying educational change in a national research center originally developed these tools, which are used for professional development (PD) in the Concerns-Based Adoption Model (CBAM). The tools have also been used for program evaluation (Hall & Hord, 2001; Roy & Hord, 2004).

Use of this tool to evaluate course syllabi can help teacher preparation leaders ensure that they emphasize proactive, preventative approaches instead of exclusive reliance on behavior reduction strategies. The IC included in the Appendix of this paper is designed for teacher preparation programs, although it can be modified as an observation tool for PD purposes.

The Collaboration for Effective Educator, Development, Accountability, and Reform (CEEDAR) Center ICs are extensions of the seven ICs originally created by the National Comprehensive Center for Teacher Quality (NCCTQ). NCCTQ professionals wrote the above description.

In contemporary instructional systems, the notion that one size does not fit all when it comes to instruction is understood and has influenced policy recommendations in special education and content-area instruction. It is not sufficient for teachers to simply present to students mathematics content based on a textbook or program and expect that all students will master the necessary concepts and skills. Instead, teachers must be knowledgeable about the mathematics they teach, understand how students successfully learn mathematics and which barriers commonly interfere with successful learning, and comprehend how assessment of student proficiency informs effective mathematics instruction. These characteristics will help teachers to be responsive to their mathematics instruction, the diverse nature of K-12 mathematics content, how students develop mathematical understandings, and students' performance data as indicators of student learning and drivers of instructional planning and programming.

The purpose of this IC was to provide state education agencies (SEAs), institutions of higher education (IHEs), and other stakeholders a tool to evaluate and provide a foundation for improving current state licensure requirements; state-, district-, and school-level PD activities; and teacher preparation in pre-K-12 mathematics, especially for students with disabilities and other struggling learners. We based the structure of this IC on three guiding principles: (a) teachers must understand and demonstrate mastery of the mathematics content they teach; (b) teachers must understand how students learn mathematics content; and (c) teachers must understand how assessment guides and informs instruction, including deciding what content to teach, how to teach the content, and how to evaluate instructional effects and adjust forthcoming instruction to refine and enhance student learning. Therefore, we organized the IC into three primary sections. Each section identifies key indicators of teachers' preparedness to promote positive mathematics outcomes for students with disabilities and other struggling learners. Section 1 features indicators that a teacher is sufficiently knowledgeable about mathematics content. Section 2 features indicators that a teacher understands characteristics of effective mathematics instruction (i.e., knows how to teach mathematics). Section 3 features indicators that a teacher understands how to use student assessment to guide instruction in mathematics. We structured Section 3 into three subsections: (a) 3.1: Deciding What to Teach, (b) 3.2: Deciding How to Teach, and (c) 3.3: Continually Evaluating Instructional Effects and Adjusting Subsequent Mathematics Instruction. For each section, we have provided a brief introduction and discussion for each related indicator.

Section 1: Knowledge of Mathematics Content

This section addresses four important areas of teacher knowledge of mathematics content related to effective mathematics instruction for students with disabilities and other struggling learners. Teachers must be knowledgeable of the grade-level and course-specific mathematics content they teach, including how the mathematics content relates to other mathematics content (e.g., previous grade level, subsequent grade level) for the purpose of advancing learning within and across grade levels and courses. The necessary indicators that teachers have mastered the mathematics content they will teach include the following:

- Demonstrate competency in and understand the underlying concepts for the mathematics content they teach or will be certified to teach.
- Demonstrate understandings of mathematical concepts and skills within and across domains (e.g., Counting and Cardinality, Operations and Algebraic Thinking) and how they interrelate and build upon one another over time (e.g., mathematics progressions).
- Know and can engage in the eight critical practices emphasized by the Common Core State Standards (CCSS, 2010) to promote mathematical understanding, reasoning, and problem solving.

- Demonstrate an understanding of effective teaching practices specific to the mastery of particular learning goals, content, and student proficiency.

Indicator 1.1: Demonstrate Competency in and Understand the Underlying Concepts for the Mathematics Content They Teach or Will Be Certified to Teach

Ma's (1999) seminal treatise on mathematics instruction in the United States is compelling evidence that teachers must have an understanding of the mathematics content they are expected to teach. Ma compared experienced and inexperienced mathematics teachers in the United States and China and asked teachers first to solve and then to explain how they would teach students to solve certain mathematics problems. Although the Chinese teachers had less advanced training (e.g., many had only high school degrees) than the United States teachers, the Chinese teachers handily outperformed the United States teachers in generating correct problem solutions. Alarmingly, more than half of United States teacher participants could not solve the problem 1 3/4 divided by 1/2.

Several key differences between the United States teachers and the Chinese teachers emerged in the Ma (1999) study. First, the Chinese teachers were more competent than the United States teachers in solving the mathematics problems. Second, the Chinese teachers had a better sense than the United States teachers of the key ideas and skills that underlie the correct solution of a given problem. In other words, the Chinese teachers articulated the key prerequisite skills and understandings and explained to students how to apply past knowledge to solve the current problem. Third, the Chinese teachers were fluent in demonstrating more than one way to solve each problem. Finally, the Chinese teachers were able to provide a mathematical proof to show why a problem solution worked; the United States teachers tended to use and encourage trial and error, which is both inefficient and does not facilitate consistently accurate understanding as a problem-solution strategy. Showing a student that a solution worked is not as effective as showing a student why a solution worked. Interestingly, the United States teachers tended to teach a standard algorithm for solving a problem (e.g., invert and multiply to solve division with fractions); however, they were unable to explain why or how such an algorithm worked. The implication of an algorithm-only teaching approach is that once students forget the algorithm, they have no understanding of how or why such an algorithm worked, so they are unlikely to know how to recreate it. The Chinese teachers explained the algorithms, but they went a step further in demonstrating for students how and why multiplying a number by its reciprocal works with demonstrations using whole-number division; converting fractions to division problems such as 1/2 = 1 divided by 2; and demonstrating with whole-number operations using mathematical strategies like inverse operations and commutative law and creating equivalence to solve for an unknown as multiple mathematical proofs of why the invert-and-multiply algorithm works. When students understand how the operation works, they can develop expectations for what a reasonable problem solution may be, which decreases errors and deepens understanding.

The Chinese teachers spent a significant amount of time establishing mastery for pre-identified essential skills (e.g., addition and subtraction 0-20, rapid composition and decomposition of tens as a prerequisite to place-value problem solving) to know in advance which ideas were the new key ideas to be established with instruction and explicitly connect the new ideas to past ideas through mathematical proofs. When the Chinese teachers selected tools to facilitate understanding, they were much more likely than the United States teachers to correctly align tool selection with the key idea to be established, and they more often than not used their understandings of past operations to establish new understandings. The implication of Ma’s (1999) work is that the Chinese teachers’ facility in problem solving and advanced mastery of the mathematical content caused them to be more effective than the United States teachers, and descriptive data made such a case.

Indicator 1.2: Demonstrate Understandings of Mathematical Concepts and Skills Within and Across Domains (e.g., Counting and Cardinality, Operations and Algebraic Thinking) and How They Interrelate and Build Upon One Another Over Time (e.g., Mathematics Progressions)

What expertise must teachers have with respect to K-12 mathematics content? Historically, this question has been difficult to answer in the United States because states and localities have independently determined the content of their school curricula. In addition, over the years, experts in mathematics and mathematics education have not always agreed. Recently, however, experts in mathematics and mathematics education have reached a greater level of consensus, and most states have adopted this consensus, which has led to CCSS (2010) in mathematics. These standards include the following 11 content domains or big ideas:

- Counting and Cardinality,
- Operations and Algebraic Thinking,
- Number and Operations in Base Ten,
- Number and Operations–Fractions,
- Measurement and Data,
- Geometry,
- Ratios and Proportional Relationships,
- The Number System,
- Expressions and Equations,
- Functions, and

- Statistics and Probability.

CCSS organize these mathematical domains in logical ways, and the content in each domain progresses and builds on the other domains to develop increasing depth of understanding and sophistication in mathematical reasoning and problem-solving skills (see Table 1). During the elementary grades, CCSS emphasize content related to the following domains: Counting and Cardinality, Operations and Algebraic Thinking, Number and Operations in Base Ten, Number and Operations–Fractions, Measurement and Data, and Geometry. Proficiency in these domains provides students with a foundation for success with middle school and high school mathematics. In middle school, the domain of Geometry receives continued emphasis, and students' understandings of the following K-5 domains are utilized to develop proficiency: Ratio and Proportional Relationships, The Number System, Expressions and Equations, Functions, and Statistics and Probability. By the time students reach high school, they have engaged in learning related to all 11 domains. At the high school level, a variety of mathematics courses (e.g., Algebra 1, Algebra 2, Geometry, Integrated Math 1, Integrated Math 2) primarily, but not exclusively, emphasize advanced concepts and skills related to the following domains: Operations and Algebraic Thinking, Geometry, The Number System, Expressions and Equations, Functions, and Statistics and Probability.

Table 1

Common Core State Standards Domains Emphasized at the Elementary School, Middle School, and High School Levels

Domain	Elementary School	Middle School	High School
Counting and Cardinality	K only		
Operations and Algebraic Thinking	Grades 1-5		Included in one or more courses
Number and Operations in Base Ten	Grades 1-5		
Number and Operations–Fractions	Grades 3-5		
Measurement and Data	Grades 1-5		
Geometry	Grades 1-5	Grades 6-8	Included in one or more courses
Ratios and Proportional Relationships		Grades 6-7	
The Number System		Grades 6-8	Included in one or more courses
Expressions and Equations		Grades 6-8	Included in one or more courses
Functions		Grade 8	Included in one or more courses
Statistics and Probability		Grades 6-8	Included in one or more courses

Note. At the high school level, a variety of mathematics courses (e.g., Algebra 1, Algebra 2, Geometry, Integrated Math 1, Integrated Math 2) primarily, but not exclusively, emphasize advanced concepts and skill related to operations and algebraic thinking, geometry, the number system, expressions and equations, functions, and statistics and probability.

Indicator 1.3: Know and Can Engage in the Eight Critical Practices Emphasized by the Common Core State Standards to Promote Mathematical Understanding, Reasoning, and Problem Solving

As illustrated in Ma's (1999) study, there is more to the content of mathematics than the mathematical concepts and skills—the "what" of mathematics. The content of mathematics also includes the "doing" of mathematics—doing those mathematical practices that promote understanding, reasoning, solving problems, making connections, representing, and communicating mathematics in sophisticated ways. Therefore, CCSS in mathematics emphasize eight mathematical practices that are critical for developing the understandings and skills exemplified by the Chinese teachers in Ma's study. The eight practices were distilled from the National Council for Teachers of Mathematics process standards (i.e., problem solving, reasoning and proof, communication, connections, and representation; NCTM, 2014b) and the National Research Council's *Adding It Up* components (i.e., adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, and productive disposition; NRC, 2001). The CCSS eight mathematical practices are as follows:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Strategically use appropriate tools.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

These eight practices serve as a framework for how students should engage in learning and doing mathematics across the K-12 curriculum. In order for teachers to effectively engage students in utilizing these mathematical practices, teachers must be able to demonstrate their abilities to engage in these practices. For more information about these eight mathematical practices, visit the Common Core State Standards Initiative website (http://www.corestandards.org).

Indicator 1.4: Demonstrate an Understanding of Effective Teaching Practices Specific to Mastery of Particular Learning Goals, Content, and Student Proficiency

Expertise in mathematics content alone has not been shown to be sufficient for establishing successful mathematical learning for students. Hattie (2009) found that subject-matter knowledge was negligibly related to student achievement (d = .09). Ma (1999) also acknowledged the necessary but insufficient role of teacher knowledge, stating, “A teacher’s subject-matter knowledge may not automatically produce promising teaching methods or new teaching conceptions . . .” (p. 38), but “. . . without solid support from subject matter knowledge, promising methods or new teaching conceptions cannot be successfully realized” (p. 38).

Slavin and Lake (2008) conducted a synthesis of experimental and quasi-experimental research on mathematics achievement and applied rigorous criteria to include studies of sufficient quality to permit meaningful conclusions (87 of 256 studies reviewed met their inclusion criteria). There were three general types of studies: (a) studies of mathematics curricula (n = 13), (b) studies of computer-based instruction (n = 36), and (c) studies designed to change the teacher-student interaction during mathematics instruction (e.g., increase teacher feedback; n = 36). In general, studies designed to change teacher-student interaction were of the highest experimental quality. Curricula effects were weak (d = .10). Computer-based instruction involved practicing mathematics facts on a computer and produced a small to moderate average effect (d = .19).

However, studies that assessed and changed teacher-student interaction during mathematics produced a moderate achievement effect that was, on average, three times the effect of the mathematics curricula (d = .33). The most powerful lesson of the research data is that content knowledge (i.e., knowing what to teach) and the science of effective instruction (i.e., knowing how to teach) are both essential prerequisites to successful learning outcomes for students in mathematics. Section 3.2—Deciding How to Teach—details how teachers can select instructional strategies to suit the content, goal of instruction, and student proficiency.

Section 2: Knowledge of How Students Learn Mathematics

Although knowledge of mathematics content is critical for teachers of mathematics to be successful, knowledge of students is also important. Teachers must be knowledgeable of how students learn mathematics as well as the common barriers that can make learning mathematics difficult for some students. Necessary indicators that teachers have mastered an understanding of how students learn include the following:

- Demonstrate understandings of how typical students' mathematical thinking develops over time for foundational concepts.
- Demonstrate understandings of common mathematical misconceptions and error patterns that represent faulty mathematical thinking.
- Demonstrate understandings of potential barriers to learning mathematics for students with disabilities.

Indicator 2.1: Demonstrate Understandings of How Typical Students' Mathematical Thinking Develops Over Time for Foundational Concepts

Teachers cannot simply present the content that appears in the adopted mathematics textbook and presume that learning will follow. Instead, teachers must provide sufficient acquisition instruction to establish understanding given the students' current levels of mathematical understanding and mastery of associated or prerequisite skills. Aligning instruction with student performance means that teachers provide mathematical explanations that rely on the concepts and skills students have already mastered. When a teacher expects children to approach mathematics in the same way that adults approach mathematics, the teacher may fail to recognize that a student's mathematical thinking is accurate (e.g., when a student uses an alternative algorithm or problem-solving strategy; Carpenter, Fennema, Franke, Levi, & Empson, 1999), or the teacher may encourage and reward the use of a procedure or process that is mathematically accurate but may not be appropriate for a student because of the student's age or lack of prior mathematical knowledge. For example, students typically build multiplicative reasoning using repeated addition before they begin using multiplicative strategies like partial products or the distributive property. While transitioning from reasoning with addition to reasoning with multiplication, students begin to utilize skip counting and the use of arrays.

A teacher who does not recognize where students are in this developmental sequence may expect students to engage in mathematical reasoning for which they are not ready. For example, expecting a student who has not yet developed multiplicative reasoning to utilize partial products to solve a two-digit by two-digit multiplication equation will likely lead to frustration and a lack of success for that student. When teachers can combine their knowledge of how students typically approach mathematics at different stages of learning and development with an assessment of student mastery of prerequisite skills and understandings, teachers can adjust and focus instruction to build upon students' established skills and understandings. States are increasingly calling for individualized education program (IEP) goals to connect to grade-level standards. Therefore, when students demonstrate gaps in mathematical knowledge and require instruction in prerequisite content, teachers must be able to connect IEP goals to current grade standards. Teachers who understand the mathematical big ideas that span grade levels can develop IEP goals that provide well-aligned instruction for the student that ultimately will make the student more capable of mastering grade-level content. For example, a third-grade student may have difficulty with the following Grade 3 CCSS standard: *Use place-value understanding to round whole numbers to the nearest 10 or 100*. A teacher who understands how the big idea of place value connects to Grade 2 standards would realize that several Grade 2 place-value standards correlate with this Grade 3 standard (e.g., *Mentally add 10 or 100 to a given number 100-900; Read and write numbers to 1000 using base-ten numerals, number names, and expanded form*). Although the IEP goal may reflect the Grade 3 standard, the teacher must understand how to utilize instruction that supports proficiency in related Grade 2 place-value standards as a means to build proficiency in the Grade 3 standard and include them as objectives for accomplishing the Grade 3 IEP goal.

All mathematics content is not equivalent from a learner perspective. For most people, some content is easier to understand than other content. Teachers must understand and anticipate challenges associated with certain mathematical concepts. The area of fractions, for example, is content with which many United States students struggle (Hoffer, Venkataraman, Hedberg, & Shagle, 2007). For some students, conventional fractions turn the meaning of numbers upside down (Krasa & Shunkwiler, 2009). Fractions are the first instructional occasion in mathematics in which the base unit is not 1 (or cannot be made to be 1), which causes numbers and operations to function differently than students have previously experienced in their mathematics instruction. For example, the numbers used in fractions do not have a natural counting sequence, and they do not have a unique number representation for each fraction. Therefore, a student who has mastered whole-number concepts understands that 4 = 4, and no other number by itself can be made to equal

4. With fractions, two values (e.g., 1/4 and 2/8) are equivalent numerical quantities, but they use different numbers. The relationship between magnitude and number is also an issue with fractions. Consider fractions less than 1 (e.g., 1/3, 1/7). The fraction with the bigger number in the denominator is smaller than the fraction with the smaller number, which runs counter to what students have learned about cardinality and ordinal position with whole numbers. When multiplying whole numbers, the product is always a greater number, and when dividing whole numbers, the quotient is always a lesser number. However, the opposite is true when multiplying and dividing fractions that are less than 1.

Because fractions often present challenges to students, teachers can anticipate and plan for this fact prior to instruction. Knowing that fractions cause confusion for most students is information teachers can use to help students to understand how fractions work, how they relate to whole numbers and whole-number operations, and how they expand students' capacities for solving mathematical problems. With such knowledge, elementary teachers can utilize the Siegler, Fazio, Bailey, and Zhou (2013) recommendations for teaching fractions such as building on students' informal understandings of sharing and proportionality and use of number lines to represent fractions. Teachers should also make adjustments to the amount of instructional time and practice opportunities they provide to students learning fractions and the explicitness of corrective feedback they provide during instruction because misunderstandings and errors are probable. Teachers should also provide more frequent monitoring of student mastery of essential fraction concepts and operations to determine whether students need continued acquisition instruction or whether students are ready for fluency building or generalization opportunities. The important point is that teachers can and should adjust the intensity of their instruction while teaching a skill or concept that is known to cause difficulty for students. Therefore, possessing knowledge of key mathematics content areas that can be especially problematic for students is an important skill to have.

Indicator 2.2: Demonstrate Understandings of Common Mathematical Misconceptions and Error Patterns That Represent Faulty Mathematical Thinking

Instructional practices that align well with students' needs produce stronger learning gains (Burns, Codding, Boice, & Lukito, 2010). Specifically, while students are learning a new skill, student performance will likely be inaccurate or incomplete, and effective instructional practices will include modeling of correct and incorrect responding, use of prompts and cues to establish correct understandings, elaborate and immediate corrective feedback to advance students' understanding, and a repetition loop such that students have an immediate opportunity to correct errors with support from the teacher (Burns, Riley-Tillman, & VanDerHeyden, 2012; Harniss, Stein, & Carnine, 2002). When a student is acquiring a new skill, the teacher must understand and make explicit how the new skill connects to existing knowledge and be able to demonstrate why a response is correct (e.g., using mathematical proof with operations the student already knows) or incorrect. To establish correct responding, the teacher must anticipate common misunderstanding and error patterns that occur while teaching a new skill. See Howell, Fox, and Morehead (1993) and Ginsburg (1987) for more information on completing an error-pattern analysis. Then, teachers can anticipate the errors before they occur, can be vigilant to and monitor for their occurrence, and can provide immediate error correction as instruction progresses. Assessing to verify that students have mastered the relevant prerequisite skills should occur, and assessment to verify student understanding should also occur during acquisition instruction. See Figure 1 for an example of a common error pattern related to two-digit addition with regrouping.

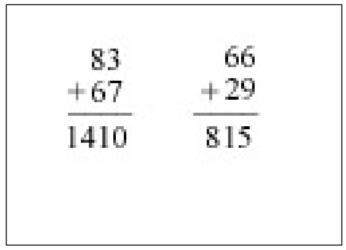

Figure 1. Common error pattern related to two-digit addition with regrouping showing lack of understanding of place value.

The sums of the ones and tens are each recorded without regrouping. This error pattern reflects a lack of understanding of place value and procedures associated with composing a higher value unit in addition (i.e., regrouping). When teachers can predict that students will have difficulty using place-value properties to solve addition, they can allocate instructional effort to reconnect the concept of place value and trading ones for tens and tens for ones to the process of adding two-digit numbers. An effective strategy may be to write each addend in expanded form, solve, and then convert back to standard notation. Teachers can monitor to ensure students can respond accurately and explain their problem-solving processes and can provide more elaborate corrective feedback about errors.

Indicator 2.3: Demonstrate Understandings of Potential Barriers to Learning Mathematics for Students With Disabilities

Students with more significant mathematics difficulties (e.g., learning disabilities) can present slightly different challenges for teachers during acquisition instruction due to disability-related learning characteristics. For example, some students with learning disabilities demonstrate difficulties with inhibition control. Poor inhibition control occurs when students have difficulty filtering out irrelevant mathematical associations (Geary, Hamson, & Hoard, 2000; Geary, Hoard, Nugent, & Byrd-Craven, 2007). For example, when recalling basic addition facts such as 4 + 2, students with learning disabilities may associate the individual digits 4 and 2 with the number that follows them in a counting sequence—in this case, 5 and 3. Therefore, students respond with the sum 8 instead of 6. Teachers who understand this can provide students exhibiting poor inhibition control with more intensive instruction (e.g., more opportunities to respond to facts that represent association difficulties, teaching learning strategies that support meaningful associations) and supports (e.g., providing wait time for responses and cue sheets that prompt use of taught memory strategies).

Students with learning disabilities can also have difficulties with visual-spatial processing and associating mathematical language with the abstract notations used to represent mathematics (i.e., numbers and symbols). For example, students with both mathematics disabilities and reading disabilities are especially at risk of mathematics difficulties (Jordan, 2007) because they have trouble with both the language and number-sense demands placed on them while learning and doing mathematics; this affects students' abilities to problem solve whether or not it occurs within the context of word problems. It is likely that students with these difficulties will need explicit instruction in mathematics vocabulary and symbols in order to establish accurate understandings and fluent performance (e.g., providing students with antecedent cues to prompt the correct operation such as highlighting the operation in the problem and offering immediate corrective feedback when students perform the wrong operation).

Other student-learning characteristics that can impact mathematics include metacognitive deficits, learned helplessness, passive approaches to learning, cultural and linguistic diversity, academic skill gaps, and mathematics anxiety (Allsopp, Kyger, & Lovin, 2007; Kersaint, Thompson, & Petkova, 2013; Miller & Mercer, 1997). Metacognitive deficits refer to difficulties students have with thinking about their thinking. Students with metacognitive deficits do not naturally apply efficient strategies while problem solving, do not monitor the effectiveness of strategies they are utilizing, and do not change ineffective strategies. Students with metacognitive deficits also do not naturally make meaningful connections among related mathematics concepts and skills. Metacognitive deficits combined with learned helplessness lead to students who approach the learning and doing of mathematics in passive ways. Learned helplessness occurs when students continually experience failure in mathematics because they lack the learning and self-regulation strategies to be successful. Over time, students believe that they cannot be successful in mathematics unless someone helps them. They have the mistaken belief that there are students who are good at mathematics and those who are not, and they believe that they are not good at mathematics. As a result, they become passive in their approaches to learning and doing mathematics. Teachers who understand these issues can emphasize explicit practices related to teaching metacognition in mathematics (e.g., thinking aloud one's thoughts as the teacher models problem solving, teaching problem-solving strategies, utilizing visuals and graphic organizers to show connections among concepts) and helping students visualize their progress by charting their performance on target concepts and skills and teaching goal setting. Students with cultural and linguistic differences can have difficulties with mathematics because the language and contextual experiences utilized in the mathematics classroom are not meaningful. The vocabulary, syntax, and semantics used in mathematics can become significant barriers for students who are culturally and linguistically diverse (CLD; see Aceves & Orosco, 2014). Examples of how mathematics teachers can address some of these issues include (a) asking students to restate what they heard to determine whether what they heard is what the teacher intended; (b) restating or paraphrasing statements made by students and asking students if the teacher restatements were what the students intended; and (c) using pictures, objects, and gestures to reinforce verbal communication. Students with disabilities often present with gaps in their understandings of mathematics due, in part, to the characteristics described in this section. Moreover, skill gaps in other academic areas (e.g., reading) can exacerbate skill gaps in mathematics because these skills are critical for success in mathematics. Indeed, mathematics can become an anxiety-provoking subject for students with disabilities for very real reasons. Teachers who understand this can decrease students' anxiety over time by utilizing effective practices during instruction that result in success for students.

Teachers must be able to anticipate, identify, and troubleshoot the difficulties that some students experience that will interfere with learning mathematics. While teaching new content and skills, teachers must attend to learner understanding and intensify instruction (i.e., provide greater antecedent supports; provide more frequent, more immediate, and more elaborate corrective feedback; provide a more subtle fading of antecedent support for correct responding; frequently monitor correct responding and understanding; and provide a higher dosage of practice opportunities; Gersten et al., 2009). Student need should drive instructional intensity, and students with disabilities are likely to require more intensive instruction (Fuchs et al., 2008; Gersten & Chard, 1999). In Section 3.2, we have provided a more extensive discussion of effective instructional practices for students with disabilities and other struggling learners.

Section 3: Assessment and Instruction: The Teaching of Mathematics

Assessment and instruction are integrated actions in the teaching of mathematics. Teachers cannot plan which skills to teach in the absence of assessment information telling them what students know and what students must learn to experience success in forthcoming instruction. In this section, we have discussed three important areas of assessment and instruction related to the effective teaching of mathematics for students with disabilities and other struggling learners; these three areas are (a) determining what to teach, (b) determining how to teach, and (c) evaluating the effects of instruction and making subsequent instructional decisions.

Subsection 3.1: Deciding What to Teach

In this subsection, we have detailed the importance of teachers deciding which mathematical concepts and skills to teach. Of course, teachers should address the pertinent grade-level mathematics standards, but deciding what to teach is much more complex than simply following a curriculum map or set of mathematics standards, especially while working with students with disabilities or other struggling learners. In determining what to teach, teachers must have working plans for the skills that students are expected to have mastered and an assessment of whether students have mastered those expected skills. Decision makers outside of the classroom may use grade- and class-wide student performance data (see Section 3.3) to evaluate the extent to which instruction meets the needs of most students. However, classroom teachers must examine grade- and class-wide data to determine the need for adjustments to core instruction, provide remedial instruction to whole classes, or identify small groups or individual students who need more intensive instruction. Necessary indicators that signify that teachers can determine which mathematical concepts and skills to teach include the following:

- Specify the sequence of expected mathematics learning outcomes and place these learning outcomes on an instructional timeline with explicit consideration of multiyear learning goals.
- Use screening assessment to determine whether systemic class-wide, grade-wide, or course-specific learning deficits exist and screening data to identify students in need of supplemental support.
- Emphasize critical areas of mathematics that are foundational to mathematics success by targeting several big ideas per grade level or course for in-depth emphasis and continuous progress monitoring across the school year.

Indicator 3.1.1: Specify the sequence of expected mathematics learning outcomes and place these learning outcomes on an instructional timeline with explicit consideration of multiyear learning goals. There has been a trend over the past two decades to streamline mathematics education and address the critique that mathematics instruction in the United States has been a mile wide and an inch deep. Toward that end, policy groups have made recommendations for identifying a honed-down set of essential skills for which sufficient mastery would create more mathematically competent students. The first outcomes of this movement in mathematics were the Principles and Standards for School Mathematics (NCTM, 2014a) and the NCTM-published (2006) curriculum focal points documents, which appear to have been highly influential to the National Mathematics Advisory Panel (U.S. Department of Education, 2008) and more recently, CCSS in mathematics (2010; see http://www.corestandards.org/Math). In our discussion about mathematics content knowledge, we have primarily focused on the 11 CCSS domains and how they are positioned across the elementary school, middle school, and high school levels. Two additional sources of content-related information— NCTM's (2006) curriculum focal points and the National Mathematics Advisory Panel's (U.S. Department of Education, 2008) algebra readiness standards—can be especially helpful while making decisions about what to teach for a class, group of students, or individual students.

Curriculum focal points provided a streamlined guide to essential learning outcomes in number and operations, data analysis, measurement, and algebra; defined the essential skills specific to each grade level; and explained how grade-level skills were connected to skills learned at earlier grade levels. The National Mathematics Advisory Panel report (U.S. Department of Education, 2008) delineated a smaller subset of three foundational concepts/skills for algebra readiness specifically including fluency with whole numbers, fluency with fractions, and areas of geometry and measurement (e.g., perimeter and area of triangles and quadrilaterals, properties of two- and three-dimensional shapes, relationships between triangles and slopes of lines). NCTM (2006) included curriculum focal points specific to algebra but did not specify a set of skills that are necessary for success in high school algebra.

A comparison of the National Mathematics Advisory Panel and curriculum focal points shows that there are more similarities than differences in terms of grade-level expectations. The National Mathematics Advisory Panel suggested that students be proficient in whole-number addition and subtraction by the end of Grade 3, which is 1 year later than the curriculum focal points document suggested. The National Mathematics Advisory Panel suggested proficient multiplication and division of whole numbers by the end of Grade 5, which was consistent with the curriculum focal points. The National Mathematics Advisory Panel and curriculum focal points documents were consistent in recommending the following: representing and comparing fractions and decimals on a number line by the end of Grade 4; comparing fractions, decimals, and percent by the end of Grade 5; adding and subtracting fractions and decimals by Grade 5; multiplying and dividing fractions and decimals by Grade 6; and solving percent, ratio, and rate problems and extending work to proportionality by the end of Grade 7. The National Mathematics Advisory Panel and the curriculum focal points included mastery of operations with positive and negative fractions and integers. There was consistency in essential skills identified in geometry and measurement between The National Mathematics Advisory Panel and the curriculum focal points; however, the National Mathematics Advisory Panel required mastery of solving perimeter and area of triangles and quadrilaterals 1 year later than the curriculum focal points suggested (i.e., Grade 5 for the National Mathematics Advisory Panel and Grade 4 for curriculum focal points), and the same was true for analyzing properties of two-dimensional shapes to solve for perimeter and area and three-dimensional shapes to solve for surface area and volume (i.e., Grade 6 for the National Mathematics Advisory Panel and Grade 5 for curriculum focal points).

CCSS were apparently influenced by earlier work to streamline learning expectations in mathematics. CCSS emphasized mastery of number, including operations, relationships between operations, and place-value understandings, by Grade 3. CCSS emphasized understanding of number and operations related to fractions by Grade 4 and understanding of decimals and the rate of decomposition in moving from left to right (or composition in moving from right to left) by Grade 5. Specifically, CCSS suggested mastery of skills including fluent addition and subtraction of whole numbers within 20 by Grade 2; fluent addition and subtraction of whole numbers within 100 by Grade 3; fluent multiplication and division of whole numbers within 100 by Grade 3; mastery of the relationship between operations of whole numbers by Grade 3 (e.g., convert multiplication problems to addition, identify the inverse operation and solve for an unknown, convert more challenging problems to easier problems using the relationship of operations); multidigit multiplication and division with mathematical explanations by Grade 4; operations with decimals by Grade 5; operations with fractions by Grade 5; and use of ratios, proportions, operations with fractions, factors, multiples, and negative numbers to solve problems by Grade 6.

In summary, there is fairly consistent overlap between the three most recent and most influential policy documents offering guidance to classroom teachers about which skills should be established by which points during a student's mathematics learning career, with CCSS expecting mastery of skills slightly earlier than the National Mathematics Advisory Panel and curriculum focal points documents (e.g., whole-number multiplication and division by Grade 3 in CCSS compared to Grade 5 in the National Mathematics Advisory Panel). CCSS are the most detailed learning standards in mathematics that have benefitted from the efforts of the National Mathematics

Advisory Panel and curriculum focal points and offer teachers a sound basis for planning instruction and verifying student mastery of essential grade-level concepts and skills.

A specific sequence of concepts and skills provides a map for teachers to follow to guide students to mathematical understanding. Because instruction will unfold across many years, it is important for all teachers to take a multiyear view of math learning and instruction. The sequence of skills provides teachers with an identified set of outcomes of their instruction and understanding of how prerequisite skills relate to current instruction.

Ma (1999) referred to the notion of skill sequences as *conceptual maps* and found that knowledgeable teachers understood what the most important prerequisite skills were for new skills and could specify the big idea that was being taught while introducing a new skill. The expected learning outcomes, in sequence, give teachers a set of skills to assess to determine if instruction is working as desired and identify which students may need additional support. The standards also provide schools and districts with a system for knowing how well students are mastering the mathematics content identified as most critical to long-term school and career success.

Indicator 3.1.2: Use screening assessment to determine whether systemic class-wide, grade-wide, and course-specific deficits exist. To effectively plan and implement instruction, teachers must take inventory of what students have mastered and which learning needs remain (e.g., mastery of prerequisite skills, continued fluency-building instruction, supported practice opportunities to apply and generalize learned skills). Taking inventory of student needs should occur at the beginning of the year and should be repeated as instruction proceeds to verify mastery of grade-level skills. Teachers can use school-wide screening data to understand the learning needs of students in their mathematics classrooms. School-wide screening in mathematics should focus on the skills identified by CCSS for a grade level. A screening task should assess skills that students must master in order for the students to benefit from grade-level instruction. Student data can be examined at the school, grade, and class levels to determine the extent to which most students are mastering the expected learning outcomes. Teachers should be proficient at evaluating screening data for mathematics for their classes and their grades. Teachers can use screening data to identify which students can benefit from more challenging instruction, which students require fluency-building instruction to reach mastery, and which students need acquisition instruction to build accurate responding and conceptual understanding. In classes where the majority of students demonstrate performance above a benchmark criterion that forecasts a satisfactory level of proficiency, teachers may identify individual students who need more intensive support. For these students, teachers may offer a small-group supplemental intervention or intensive individualized intervention to ensure students reach mastery. However, if screening data in mathematics demonstrate widespread deficiencies across multiple classes in a grade level, then grade-level teachers must consider grade-level decisions about how to adapt the overall curriculum and/or instructional practices to better meet the needs of their students. Formative assessment, when effectively used in this way, is one of the most powerful instructional tools to advance student learning (Fuchs & Fuchs, 1986; Hattie, 2009; Yeh, 2007). Solutions may include utilizing a different mathematics core text/program, identifying an appropriate supplemental text or program that targets the students' areas of need, or implementing an instructional practice or set of practices that focus on an area of need (e.g., incorporating a structured peer tutoring process that provides students multiple opportunities to practice and receive immediate feedback for the purpose of building proficiency/fluency for certain prerequisite skills; Fuchs, Fuchs, Mathes, & Simmons, 1997; Greenwood, 1991; Slavin & Lake, 2008; VanDerHeyden, McLaughlin, Algina, & Snyder, 2012).

Indicator 3.1.3: Emphasize critical areas of mathematics foundational to mathematics success by targeting several big ideas per grade level or course for in-depth emphasis and continuous progress monitoring across the school year. NCTM's (2006) curriculum focal points provides teachers with mathematical concepts and skills (i.e., big ideas) that are foundational within and across grade levels. The National Mathematics Advisory Panel's (U.S. Department of Education, 2008) algebra readiness standards delineate the concepts and skills that are most important for student success in Algebra. For students with disabilities and other struggling learners who will likely need more intensive instruction, the foundational big ideas that structure curriculum focal points, the algebra readiness standards, and CCSS can provide guidance for which concepts and skills should be the focus of assessment for the purpose of informing instructional decisions about what to teach. Although each set of standards slightly differs in scope and purpose, there is more overlap than difference in the concepts and skills identified as the most important outcomes of instruction at given grade levels. Thus, the learning standards, and especially the most recent set of CCSS in mathematics, are invaluable resources for designing intensive mathematics interventions. For example, if the goal of a fifth- and sixth-grade intervention were to prepare students for success in Algebra 1 and Algebra 2, then utilizing the algebra readiness skill recommendations from the National Mathematics Advisory Panel (U.S. Department of Education, 2008) to ensure mastery of foundational skills and develop remedial interventions would be useful. For younger students, the curriculum focal points can provide educators with the foundational concepts and skills relevant to important mathematics content domains. Number-sense development is an example because of its importance to later mathematical success (Griffin, 2004; Griffin, Case, & Siegler, 1994). The curriculum focal points or CCSS surrounding number and operations and algebra could be used to identify content-related targets for assessment and intervention for these students.

Subsection 3.2: Deciding How to Teach

This subsection features indicators of effective mathematics instruction for students with disabilities and other struggling learners. Teacher educators and researchers often approach mathematics education from a theoretical or philosophical lens, which can shape their perspectives about how mathematics should be taught and learned (Parmer & Cawley, 1997). Two different approaches to mathematics instruction often discussed in the literature relate to teacher-directed instructional practices and student-directed, student-centered instructional practices. Identifying a set of practices that represent one or more philosophical approach that should be utilized above others is cumbersome given the complex nature of mathematics and the diversity of K-12 learners. The limited empirical research base that links practices to positive student outcomes compounds the issue. Members of the National Mathematics Advisory Panel (U.S. Department of Education, 2008) clearly emphasized that there is not one approach that is effective for all learners (i.e., teacher directed vs. student centered). Effective teachers should select evidence-based strategies that have been shown to work, but teachers must flexibly adapt their instructional approaches based on the needs of students and the mathematics content they are teaching (Burns et al., 2012). Indeed, relying on the textbook alone to guide instruction is not sufficient (Slavin & Lake, 2008). Instead, teachers should follow CCSS learning standards; consider student data to verify that students have mastered prerequisite skills (Foegen, Jiban, & Deno, 2007); introduce new ideas in ways that ensure conceptual understanding as we have defined it in this paper (Wu, 1999); make salient for students how the new skill can be used to accurately solve problems; build fluency for skills that students can independently and accurately perform; and provide opportunities to generalize (e.g., creating equivalent quantities, solving for an unknown, undoing an operation; Bryant et al., 2011). Given the current research base, the indicators and related practices included in this section have excellent potential for helping teachers improve mathematics outcomes for students with disabilities and other struggling learners. Indicators should not be considered to include all potentially effective practices for students generally or students with disabilities specifically (Hattie, 2009). Necessary indicators that a teacher knows how to teach mathematics include the following:

- Incorporates activities to develop conceptual understanding.
- Provides sufficient opportunities to build fluency and generalization in applying mathematical concepts and skills.
- Utilizes explicit systematic instructional practices that support mathematical learning gains for students who require more intensive instruction.

Indicator 3.2.1: Incorporates activities to develop conceptual understanding. While a teacher is introducing a new skill, the teacher should (a) be clear about which skills and understandings must precede understanding of the new concept, (b) determine whether students have mastered the prerequisite concepts and skills, (c) introduce the new skill using existing knowledge to explain how a problem solution works, and (d) understand which new concepts and skills can build upon the new knowledge in future instruction. Fluency in prerequisite skills forecasts successful learning of related future skills, especially when the new skill is introduced using high-quality acquisition instruction strategies. While delivering instruction, it is not enough to simply provide the rule, model the solution, and encourage students to memorize the steps in the procedure or algorithm (Wu, 1999). A teacher must be able to model multiple ways to solve a problem, explain and mathematically demonstrate how one solution works while another does not, and directly and explicitly teach the algorithm.

Conceptual fluency is the ability to solve a problem in multiple ways, explain why a given solution works, explain how the concept is connected to other mathematical concepts and skills, and estimate correct problem solutions for related problems. Conceptual fluency is distinct from a halting, incomplete explanation or one that simply repeats the algorithm because students will forget algorithms that are not understood, and students will have no way of reasoning their way to a correct problem solution that is at odds with the very logical and coherent natural structure of mathematics. When students understand the big ideas (e.g., the relationship between addition and subtraction), they do not need to rely on memorized tricks to solve problems.

Establishing conceptual understanding does not mean that students should be taught only easy and easy-to-visualize problems. Establishing conceptual understanding means that teachers use mathematical proofs to demonstrate why an error is an error (i.e., how it interferes with accurate problem solutions) and which logical strategies can be applied to find the correct problem solution using what students already know. Wu (1999) masterfully explained the fallacy of shying away from algorithms in the name of advancing conceptual understanding stating:

> The resistance that some math educators (and therefore teachers) have to explicitly teaching children the standard algorithms may arise from not knowing the coherent structure that underlies these algorithms: the essence of all four standard algorithms is the reduction of any whole number computation to the computation of single-digit numbers. (p. 9)

Wu (1999) maintained that it is a misguided effort to teach students to solve only problems they can draw or visualize and then encourage trial-and-error learning as the basis for conceptual understanding. Instead, teachers should use single-digit operations and students' understanding of these computations to explain how to solve more complicated problems. In Ma's (1999) study, for example, one highly competent Chinese teacher demonstrated why subtracting a three-digit number from a three-digit number from right to left is not necessary but is ultimately more efficient during required regrouping (see Figure 2).

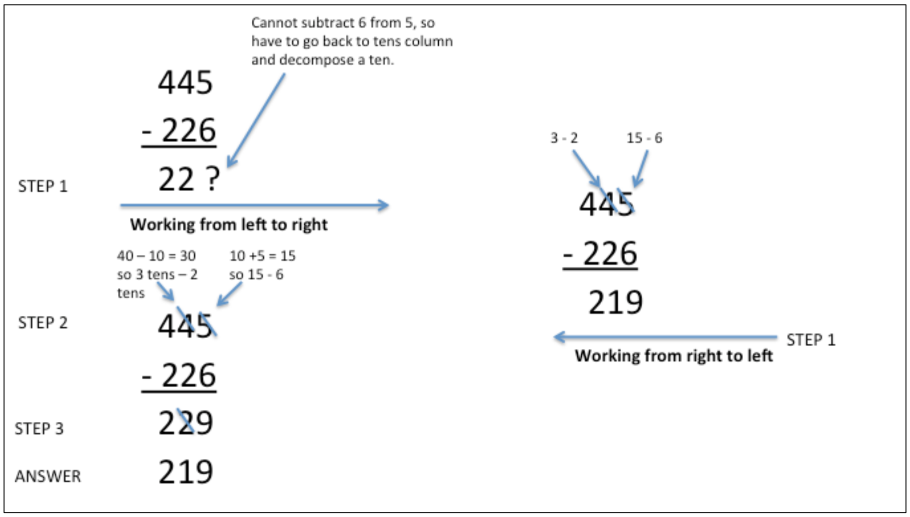

Figure 2. Solving a subtraction problem from left to right and then backing up to decompose a higher value unit and erase and change the digit that was already written in the solution space.

Similarly, while teaching the process for multidigit multiplication, a highly competent teacher may show that it is not necessary to multiply from right to left if the place-value properties are maintained in the products that will be added together for the final solution. Working from left to right is inefficient, however, and causes the problem-solver to back up and change solutions with required regrouping (e.g., composing a higher value unit; see Figure 3).

423	423	423	423
x 225	x 225	x 225	x 225
2115	8460	84600	80000 + 4000 + 600
8460	2115	8460	8000 + 400 + 60
+ 84600	+ 84600	+ 2115	+ 2000 + 100 + 15
95175	95175	95175	80000 + 14000 + 1100 + 75
			95175

Figure 3. Example of why working from left to right is inefficient and causes the problem solver to back up and change solutions with required regrouping (e.g., composing a higher value unit).

Teaching multidigit multiplication, as demonstrated in Figure 3, makes the place-value properties that undergird the algorithmic solution explicit, which makes the algorithm sensible to students; multidigit multiplication can also be more easily remembered or re-created if forgotten.

Deciding which skills to teach requires beginning with the standards and developing a plan for ensuring prerequisite concepts and skills are intact, ensuring the new concept/skill is established, and verifying students are ready for the new concepts and skills that will follow. Table 2 features examples for five skills.

Table 2

Sequencing of Concepts and Skills for Lesson Planning

Skills	Critical Big Idea or New Understanding	Prerequisite Skills	Future Understandings
Addition with regrouping	Understanding of base-ten system or place-value properties and decomposing a higher value unit	Addition 0-20 Composition of tens and hundreds	Multidigit multiplication measurement Addition with decimals
Multidigit multiplication	Sum of partial products using expanded notation and place-value properties and an understanding that it is more efficient, but not necessary, to work from right to left while solving	Addition 0-20 Multiplication 0-9 Place-value properties (e.g., 542 x 31 is 500 x 31 plus 40 x 31 plus 2 x 31)	Multidigit multiplication with decimals
Division	Rapid identification of unknown factors and understanding division as an operation that can be undone with multiplication	Multiplication 0-9	Creating equivalence between quantities Solving for an unknown with whole numbers and fractions Finding a least common denominator Finding the greatest factor to simplify a fraction
Fraction	First time base unit is not "one" Rapid identification of quantity of fraction on a number line	Mastery of basic operations (i.e., addition, subtraction, multiplication, and division) Ordinal understanding with whole numbers	Operations with fractions Operations with percentages and ratios

Skills	Critical Big Idea or New Understanding	Prerequisite Skills	Future Understandings
	Creating equivalent quantities using different and same denominators Quantity estimation for sums, differences, products, and quotients		

Indicator 3.2.2: Provides sufficient opportunities for students to build fluency and generalization applying mathematical concepts and skills. Once concepts and skills have been established, teachers should provide students with instruction designed to build fluency. Fluent performance represents more advanced mastery of concepts and skills, forecasts retention of the learned concepts and skills over time, and forecasts the ability to apply or adapt the concepts and skills to solve novel and more complex problems (Johnson & Layng, 1992). Two students may score 100% accuracy on a mathematical task, but one may be much less proficient and may need different instruction than the other. Imagine a student who can accurately provide a solution but must draw and count hash marks, double check the answer, and provide a halting and incomplete explanation for the solution. Now, imagine a student who answers the problem without hesitation and when asked to explain it can explain the solution and possibly even solve the problem a different way. The second student is more proficient, and fluency will reflect this.

Fluency moves beyond the accuracy of responding, adds a timed dimension to the response (Johnson & Layng, 1992), and has been defined as *accuracy plus speed* (Binder, 1996). In mathematics, computational and procedural fluency can be measured by identifying the digits correct per unit of time on a content-controlled task. There is a strong consensus that conceptual understanding and fluent skill performance are intertwined and bidirectional such that one begets the other and vice versa, and excellent instruction emphasizes both (U.S. Department of Education, 2008). In 2001, the National Resource Council (NRC, 2001), the precursor to the National Mathematics Advisory Panel, stated,

> Procedural fluency and conceptual understanding are often seen as competing for attention in school mathematics. But pitting skill against understanding creates a false dichotomy. As we noted earlier, the two are interwoven. Understanding makes learning skills easier, less susceptible to common errors, and less prone to forgetting. By the same token, a certain level of skill is required to learn many mathematical concepts with understanding, and using procedures can help strengthen and develop that understanding. (p. 122)

Fluently performing a skill reduces the difficulty associated with solving multistep problems, solving application problems, and learning new related content in the future. However, fluency-building instruction is often overlooked or de-emphasized in mathematics instruction at a substantial cost to student achievement (Loveless, 2003). Fluency-building instruction is appropriate for students who can accurately respond without adult support and includes strategies like assessing/monitoring fluent skill performance; setting goals for more fluent performance over time; and frequent, uninterrupted practice intervals with delayed corrective feedback.

Because fluency-building instruction commonly receives inadequate focus or is ineffectively carried out, it is a ripe target for mathematics performance improvement in many schools. Fluency-building instruction should occur daily and should target a skill that students know how to accurately perform. Many readers may think of drill-and-kill worksheets as representing fluency-building instruction, and this association is not accurate in most places. Worksheet practice is often not optimally effective because (a) the skill is not selected based upon student performance, (b) skill difficulty and content is not advanced based on student performance gains, (c) corrective feedback is not provided immediately following the practice interval, (d) the problems to which the student is exposed are always presented in the same order using the same format, (e) goals and positive reinforcement are not provided for more fluent performance, and (f) performance is not tracked at all to show growth. When fluency-building strategies are aligned with student proficiency and provided in a high-quality way, achievement gains are observed (Fuchs et al., 1997; VanDerHeyden & Burns, 2005; VanDerHeyden et al., Snyder, 2012).

For students who struggle to master essential grade-level mathematics skills, daily fluency-building instruction on prerequisite concepts and skills at students' instructional levels is a powerful method to close achievement and performance gaps (VanDerHeyden et al., 2012). For students who struggle, small-group supplemental instruction is ideal for fluency-building instruction on prerequisite and foundational skills as a powerful complement to core instruction (Bryant et al., 2011). Students with disabilities who persistently struggle may require more trials to reach proficiency, and these trials can be provided with multitiered instruction.

Once students can independently and fluently respond, teachers should provide opportunities to generalize the skill. Generalization problems can include conversion of problems to easier problems, creating equivalent problem solutions, solving for unknowns, and solving word problems. Students need multiple opportunities to apply concepts and skills with which they have become fluent within problem-solving situations. Problem solving has to do with much more than solving word problems. It involves finding solutions to applied problems that are situated in different contexts ranging from tangible to abstract. Applied problem solving requires students to utilize both conceptual and procedural understandings within or across multiple mathematical domains or big ideas (e.g., solving for an unknown to solve an angle, applying a known rate to solve for an unknown unit of time). Problem solving also requires students to apply strategies that organize their thinking about the problem and how to solve it.

When students can generalize individual skill sets and adapt their mathematical understandings to build new understandings, they become effective mathematical problem solvers. In contrast to students learning isolated sets of mathematics concepts and skills for a grade level or course, this type of mathematical thinking and skill is the cornerstone of CCSS in mathematics. The foundation for this mathematical competency is built by providing all students with supported opportunities to solve problems and discuss and justify problem solutions every day. In many mathematics homework assignments, there are a great number of computation problems and only one or two thinking problems, which does not provide adequate practice opportunities for most students. Teachers should provide sufficient practice solving applied problems and asking students to think out loud or teach someone else how to solve problems, including having students post their solutions and utilizing student postings as a means for modeling and providing feedback (Ball, 2011); this should begin during kindergarten and should gradually require more sophisticated explanations as learning progresses.

Indicator 3.2.3: Utilizes explicit systematic instructional practices that support mathematical learning gains for students who require more intensive instruction. Students with disabilities and other struggling learners will likely require mathematics instruction that is more intensive in nature than mathematics instruction for their peers who do not struggle. Therefore, it is important that teachers of mathematics understand which types of instructional practices provide the learning supports that students with disabilities and other struggling learners will need in more intensive mathematics instruction contexts.

Explicit instruction that is systematically planned and implemented based on student performance data is a key foundation to successful mathematics instruction for students struggling to proficiently learn mathematics (Gersten et al., 2009). Indicators 3.2.1 and 3.2.2 highlighted the importance of teachers emphasizing conceptual understanding and fluency building in their instruction. Teachers should maintain an emphasis on these important areas for students in need of more intensive support by utilizing systematic and explicit practices. Therefore, teachers should have strong foundations in core components of mathematics instruction that are systematic and explicit in nature. The systematic aspect of this instruction ensures that clear and appropriate mathematics learning objectives guide instruction; that teachers provide students with opportunities to fully understand and become proficient with newly presented concepts and skills, including supporting students' generalizations of concepts and skills to new contexts; that there are appropriate methods for evaluating student learning and communicating this to students; and that student performance data is used to make decisions for subsequent instruction. This type of instruction is characterized by the more frequent use of salient cues and prompts for correct responses to mathematical tasks; sophisticated prompt fading techniques (i.e., instructional scaffolding); opportunities to respond over a longer period of time (Greenwood, 1991); immediate and/or elaborate corrective feedback (Krohn, Skinner, Fuller, & Greear, 2012; Poncy, Fontenelle, & Skinner, 2013); frequent progress monitoring; gradual increases in task difficulty; and guided practice solving related applied problems (Kavale & Forness, 1999; Swanson, 1999).

Within a systematic instruction frame, the utilization of explicit teaching practices can support students' understandings of targeted mathematics concepts and skills. Teachers should understand that explicit teaching is not akin to telling students what to do and how to do it. Explicit mathematics teaching is that which provides students with clear and transparent pathways to understanding mathematical concepts and skills by connecting new mathematics concepts and skills to previously learned concepts and skills; situating mathematics concepts and skills within meaningful contexts; providing models of target concepts and skills and utilizing cueing techniques to highlight the most essential characteristics (i.e., parts); placing an emphasis on teaching the thinking (e.g., mathematical reasoning) behind the procedures of mathematics (e.g., algorithms); providing students with multiple opportunities to apply newly learned mathematics concepts and skills, including opportunities to express their understandings in different ways (e.g., drawing, tables/graphs, concrete materials, graphic organizers, verbalization); providing feedback on students' successes and areas for improvement; and teaching self-evaluation and goal setting (i.e., self-regulation; Allsopp et al., 2007).

Several recent syntheses of research and related studies have emphasized the positive effect that explicit systematic mathematics instruction has on promoting positive outcomes for students with disabilities and other struggling learners. In a synthesis of research on elementary and middle school mathematics practices, Gersten and colleagues (2009) recommended several explicit systematic practices that promote positive mathematics problem-solving outcomes for students with disabilities and other struggling learners. One example is explicit mathematics instruction that focuses on problem solving in which teachers model efficient processes/strategies for problem solving, incorporates verbalization of teachers' thought processes as teachers model, provides guided practice in which teachers coach students to assume more and more responsibility for problem solving on their own with corrective feedback, utilizes cumulative review in which teachers connect previously learned concepts and skills to what students are currently learning, and engages students in thinking about how the concepts relate. Another example from the Gersten and colleagues synthesis is related to solving word problems in which explicit instruction centers on teaching students to identify the structure of word problem types and how to discriminate superficial from substantive information for determining when to utilize problem-solving strategies students have already learned. A synthesis of research by Woodward and colleagues (2012) provided additional examples of effective explicit instruction practices for problem solving. One recommended practice is teaching students to monitor and reflect on their problem-solving efforts. This includes providing students with prompts that cue them to monitor and reflect as they engage in problem solving, modeling how to monitor and reflect on the problem-solving process, and coaching students through teacher-student dialogue to use their own thinking about problem solving to build monitoring and reflection skills. Fuchs and colleagues (2003) reported the positive effects of teaching students self-regulatory learning (SRL) strategies to enhance problem solving. This entails explicitly teaching students strategies for goal setting and self-evaluation in conjunction with instruction to transfer learned problem-solving strategies to different problem-solving situations (e.g., students broadening the categories in which they grouped problems requiring the same solution methods, students searching for novel problems that fit these broad categories).

The use of visuals to represent mathematical ideas is a primary component of explicit instruction and has been utilized in a variety of ways to promote positive mathematics outcomes for students with disabilities and other struggling learners. Woodward and colleagues (2012) described how the use of visual representations (e.g., strip diagrams, percent bars, schematic diagrams) for problem solving can promote positive outcomes. In visual representation, teachers select representations that are appropriate for students and the problems they are solving, they utilize think-alouds and discussions to model the use of visual representations for problem solving, and they model and coach students about how to convert their visual representations into mathematical notations. The use of visual representations of mathematical ideas (e.g., manipulatives, drawings, graphs, number lines) was also found to be an effective practice in the Gersten and colleagues (2009) synthesis. Specifically, the use of concrete materials was found to be effective when the use of visuals themselves was not sufficient. One explicit systematic practice that has resulted in positive outcomes in multiple studies (e.g., Bryant, Bryant, Gersten, Scammacca, & Chavez, 2008; Flores, 2010; Fuchs et al., 2005; Gersten et al., 2009; Maccini & Hughs, 2000; Witzel, 2005) is the utilization of a concrete-representational-abstract (CRA) sequence of instruction in which students are provided systematic and scaffolded opportunities to experience models and practice applying mathematics concepts and skills by using materials (i.e., concrete level), drawing representations (i.e., representational), and, finally, using mathematical numbers and symbols only (i.e., abstract level). Another evidence-based practice (EBP) that has resulted in positive mathematics outcomes for students with disabilities and other struggling learners includes anchoring problem solving in authentic and relevant contexts (Bottge, Heinrichs, Chan, & Serlin, 2001; Bottge, Heinrichs, Mehta, & Hung, 2002; Bottge et al., 2004; Bottge, Rueda, & Skivington, 2006). In related studies, problem solving was explicitly connected to problems situated in video centered vignettes/stories.

In this indicator, we have emphasized the importance of explicit systematic mathematics instruction for students in need of intensive support. Teachers must understand and be able to apply explicit systematic mathematics practices for students with disabilities and other struggling learners in order to improve mathematics outcomes. However, it should not be construed that the implementation of explicit systematic mathematics instruction should occur in a vacuum that is void of other important mathematics practices such as the eight essential mathematics practices suggested by CCSS and described under Indicator 1.3 or other high-leverage practices (Ball, Sleep, Boerst, & Bass, 2009). Students with disabilities and other struggling learners must engage in doing mathematics in meaningful ways that will allow them to develop critical thinking and problem-solving skills, and the eight essential mathematical practices suggested by CCSS provide teachers a structure for doing this. Teachers must be able to utilize explicit systematic mathematics instruction to assist students in need of intensive support to successfully engage in the eight essential mathematical practices as well as other practices that promote reasoning, problem solving, strategic competence, and self-monitoring. Indeed, the focus of many of the evidence-based explicit systematic mathematics instructional practices described in this indicator relate to the development of these types of skills and dispositions.

Subsection 3.3: Continually Evaluates Instructional Effects and Adjusts Subsequent Mathematics Instruction

In Section 3.1.2, we discussed the importance of teachers (i.e., grade- and department-level teams) using mathematics screening data at the beginning of the school year to determine whether students are demonstrating widespread difficulties and making grade- and class-level adjustments to the curriculum and practices. This subsection highlights the importance of continually evaluating the effects of mathematics instruction and adjusting instruction during the school year when needed at the grade, class or course, and individual student levels. When teachers, as part of grade- or department-level teams, are adept at evaluating assessment data for students across classes, they can verify that the majority of students are on track to meet year-end learning objectives. When the majority of students are not on track to meet year-end learning objectives, the grade-level team can plan, implement, and evaluate corrective actions. At the class or course level, the screening/continual progress-monitoring data can be used to identify students who are meeting or exceeding benchmark criteria and students who are not and identify and implement strategies to facilitate learning gains for all students. At the individual student level, teachers must use formative mathematics assessments to identify and diagnose student misconceptions, faulty mathematical thinking, and barriers to mathematical proficiency. Knowing how to use assessment data to guide instruction is one of the most reliable and effective ways for teachers to improve mathematics achievement (Hattie, 2009). While working with struggling students, a teacher's use of assessment data to guide instruction is necessary to prevent the loss of instructional time on ineffective strategies so that resources can be used to the greatest effect possible. Even well-planned and implemented instruction that appears to adhere to effective practice sometimes does not lead to positive learning outcomes for students, especially students with disabilities and other struggling learners. When the effects of mathematics instruction are not regularly evaluated during the school year, gaps in knowledge can occur for students, making it more difficult for them over time to develop accurate understandings of connected mathematical ideas. When teachers understand which concepts and skills are most essential to mathematics success at their grade levels, as well as before and after their grade levels, then they are better able to pinpoint the most critical areas while evaluating performance data and engaging in formative assessments. Necessary indicators that a teacher knows how to evaluate the effects of mathematics instruction and adjust subsequent instruction include the following:

- Engages in routine monitoring of student mastery of key mathematics concepts and skills at the grade and classroom levels to guide instructional decisions.
- Engages in routine monitoring of mastery of key mathematics concepts and skills at the individual student level to guide instructional decisions.

Indicator 3.3.1: Engages in routine monitoring of students' mastery of key mathematics concepts and skills at the student, classroom, and grade levels to guide instructional decisions. It is not sufficient to present content and presume that most students will learn it. There must be a feedback loop to monitor students' learning and guide teachers' instructional efforts, and this feedback must come from the students. Instruction should begin with student assessment of mastery of essential learning objectives. Figure 4 shows the performance of a fourth-grade class on a multiplication facts 0-9 assessment. Multiplication 0-9 is a skill that students are expected to have mastered according to CCSS by Grade 3, and it is a prerequisite skill to many of the skills that students must learn at Grade 4. Hence, verifying student mastery of multiplication 0-9 is important to guide instruction and ensure student success.

In Figure 4, every bar represents a student's performance on the multiplication facts 0-9 assessment in a single classroom at the beginning of the school year. Scores less than 40 digits correct per 2 min reflect frustration-level performance and risk for mathematics failure. Scores between 40 and 79 digits correct per 2 minutes reflect instructional-level performance. Scores greater than 80 digits correct per 2 minutes indicate mastery and forecast the ability to retain and use the skill to solve novel and more complex operations, which are required at Grade 4. In this class, no student performed at the mastery level, meaning most of these students will struggle with more complex operations including multidigit multiplication, division, and multiplication of decimals. These students will also struggle to understand division as finding an unknown factor, which is an important prerequisite to working with fractions. Hence, Figure 4 demonstrates a class-level problem in mathematics.

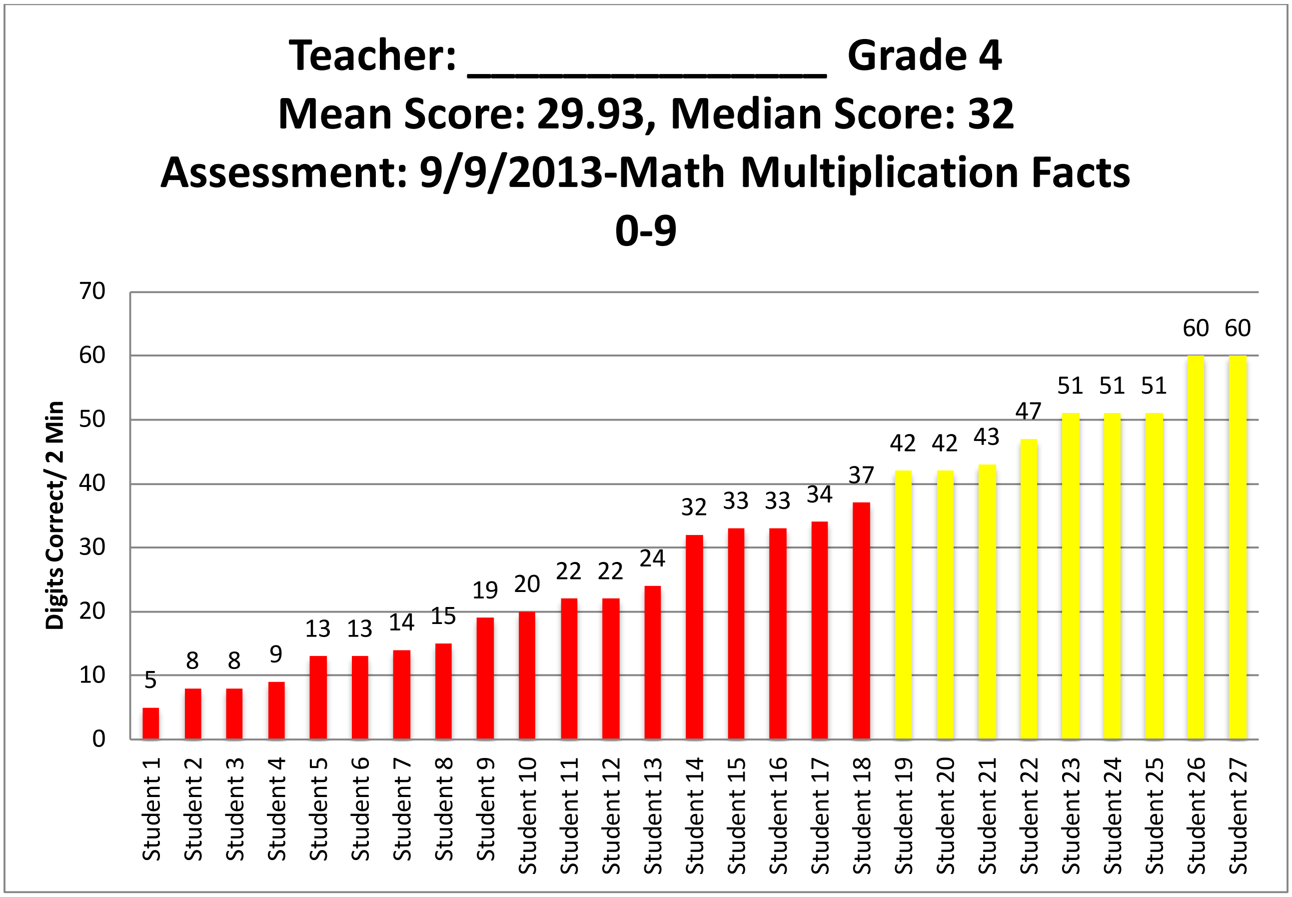

Figure 4. Example of a class-level problem in mathematics.

When a class-level problem in mathematics is detected, the teacher should first step back and consider whether there is a grade- or school-level problem in mathematics. Although not all teachers will be part of school-level data evaluation teams, it is important for all teachers to understand differences between systemic issues relative to the mathematics performance of their students and individual classroom or individual student issues. This is because treating systemic problems at the grade or school level versus an individual student level is not only a more efficient solution, but also a more effective solution. Individual interventions that are integrated within flawed systems have a low probability of success. It is important that teachers on the front lines understand this and can recognize the differences. When teachers possess such understanding, they will likely buy into systemic changes in the mathematics curriculum and related practices because they will have a clear rationale for why changes are needed. With this insight, it is likely they will more effectively implement changes.

In the Figure 4 example, two fatal flaws would compromise the potential for success of intervening at the individual level rather than at a systems level. First, the overwhelming number of required individual interventions would reduce the resources and capacity to deliver interventions well. Second, when most students in a class are low performing, the potential for any measurement tool to accurately identify individual students for necessary intervention will be weak and technically inadequate. Because teachers are the ones who will implement the mathematics interventions, they must have this systems-level perspective and understanding in order to help building administrators quickly determine when an intervention does not address an issue that is more systemic rather than individual in nature. Teachers play a central role in identifying students in need of additional supports and identification of individual students in need of more intensive supports. Teachers are critical to helping decision teams identify and repair class-level learning problems prior to conducting assessment or intervention with individual students (Kovaleski, VanDerHeyden, & Shapiro, 2013).

In Figure 5, the data were examined for the entire fourth grade from which the class in Figure 4 came. In Figure 5, each bar represents a class's performance on the screening measure; the y-axis indicates the percentage of students at risk by class. Teacher 1 had 67% of students at risk. In all fourth-grade classrooms, more than 50% of students were at risk of mathematics failure.

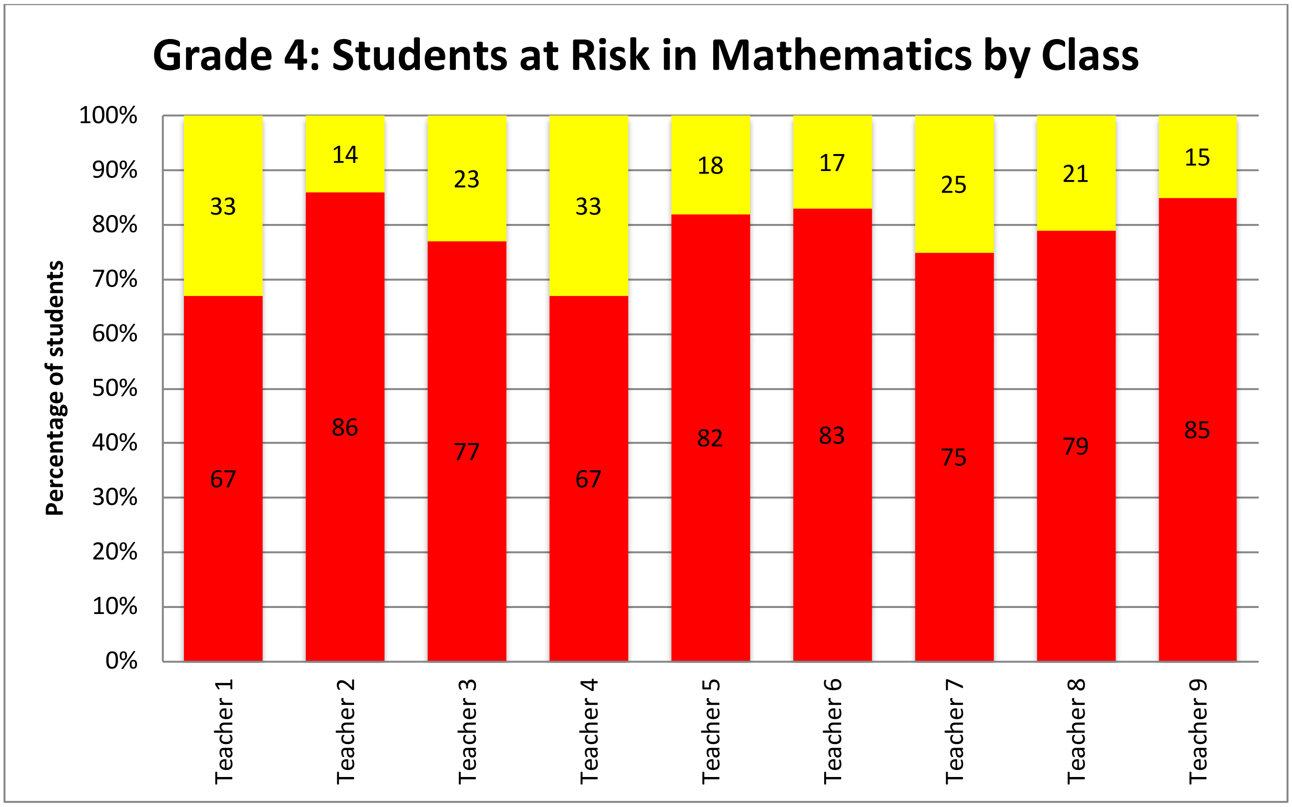

Figure 5. Entire fourth-grade data from which the class in Figure 4 came.

In this case of a grade-level problem, it makes sense to look further and identify whether multiple grades at the school are experiencing similar patterns of risk. These data should cause the grade-level team to examine the adequacy of core instruction and take corrective actions such as (a) verifying teachers' understandings of which skills are expected across grade levels, (b) specifying a calendar of instruction that effectively paces learning across the school year and across grade levels, (c) verifying that excellent acquisition and fluency-building instruction is occurring in each classroom, and (d) initiating progress-monitoring measurement to verify that corrective actions improve learning outcomes over time. The grade-level team may determine that a class-level intervention supplement in each classroom is needed; whichever corrective action is chosen, there must be a feedback loop to the grade-level team and building administrators to verify that the solution is working. Follow-up screening data are ideal for examining risk reductions across classrooms over time.

In Figure 6, the percentage of students at risk by class is shown for fall and winter for each fourth-grade teacher. In all cases, the percentage of students at risk for mathematics failure substantially declined, indicating that the intervention was a good investment of time and resources and should be continued.

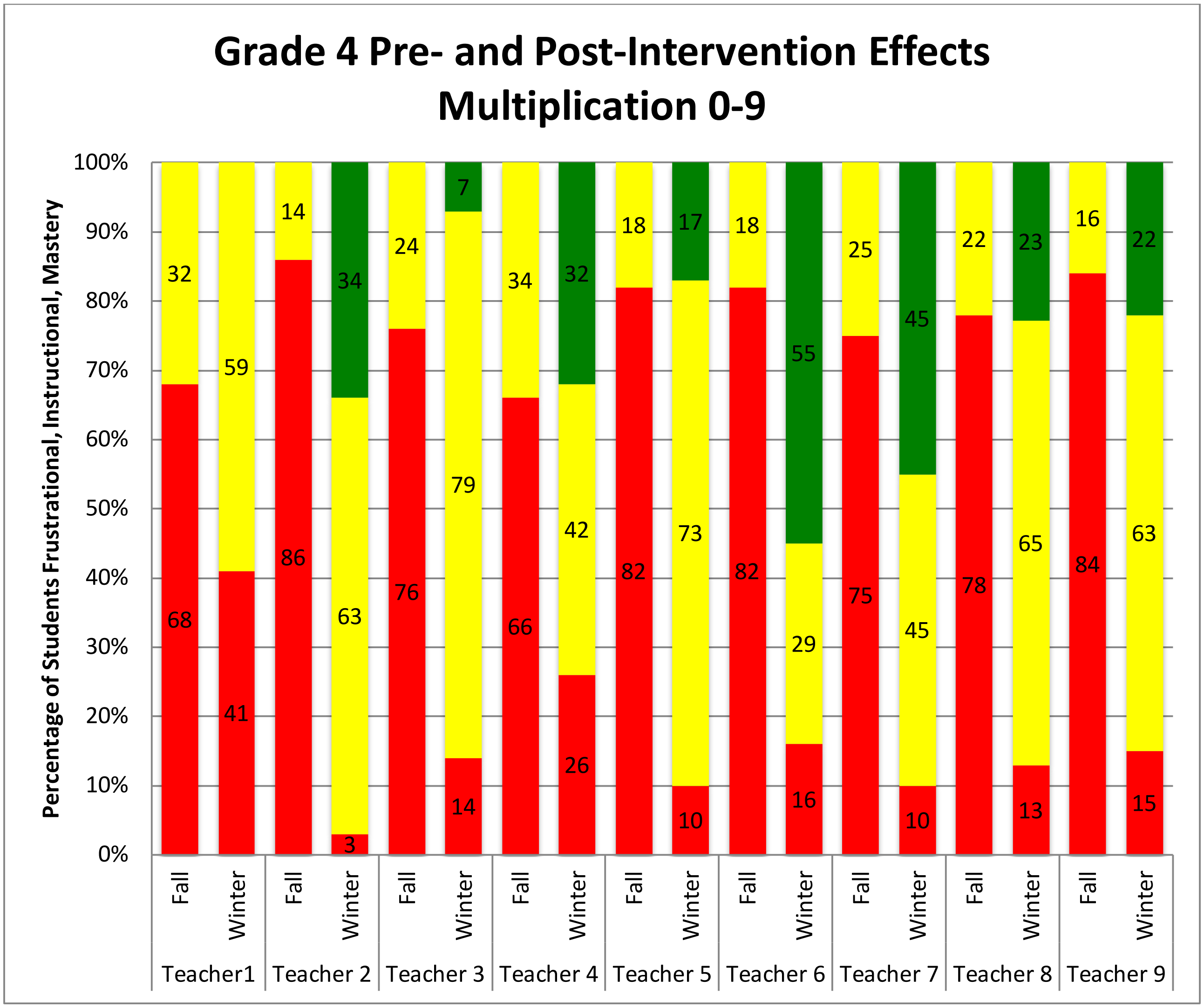

Figure 6. Follow-up screening data showing the percentage of students at risk by class for each teacher for fall and winter.

In this case, the decision by the grade-level team to implement a class-level supplemental intervention in each class resulted in success. However, the decision and result were informed by data. Without such a loop of performance feedback regarding instructional decisions, resulting interventions or changes in instruction are likely to fail (Noell et al., 2005). Performance feedback involves tracking student performance, and, at times, it may require providing a teacher additional support (e.g., in-class coaching or support to the teacher in the form of an ESE teacher or intervention specialist who may collaboratively work with the classroom teacher to address students' needs). When teachers understand the bigger picture about why certain adaptions or additional supports are instituted in their classrooms, it is likely they will be less inclined to take the changes personally and more empowered to work with the team to improve student outcomes. For example, coaching sessions are an important opportunity to troubleshoot instruction and ensure learning gains for students (Witt, Noell, LaFleur, & Mortenson, 1997), which is a win-win for teachers and students. Figure 7 shows such a situation in which data for Teachers 1-4 illustrate a need for in-class coaching and support.

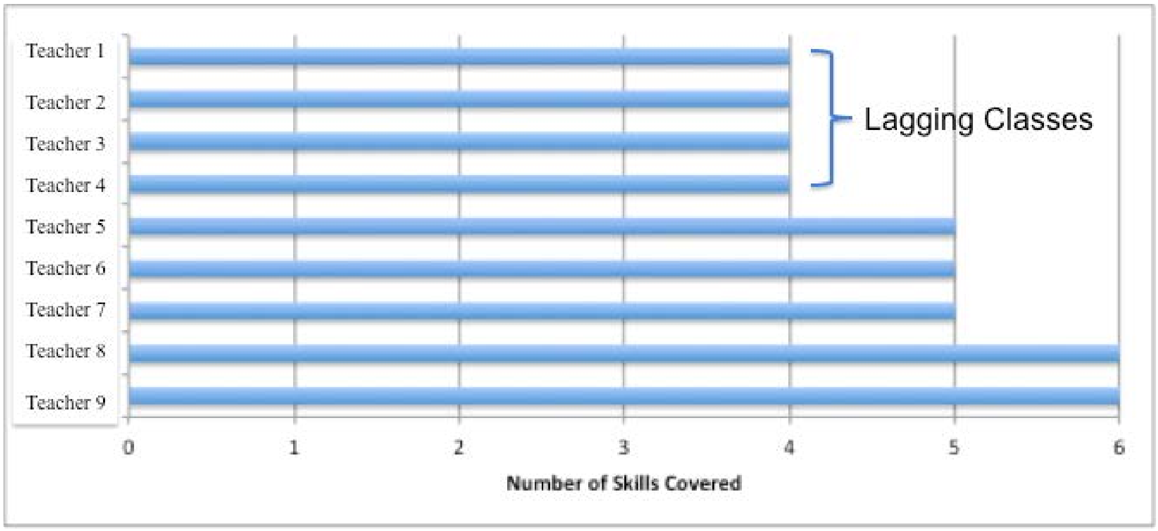

Figure 7. Illustration demonstrating that Teachers 1-4 need in-class coaching and support. Figure reprinted with permission from the National Center for Learning Disabilities (http://www.ncld.org/).

Indicator 3.3.2: Engages in routine monitoring of mastery of key mathematics concepts and skills at the individual student level to guide instructional decisions. Mathematics performance data should also be utilized in systematic ways to intervene at the individual student level. At the individual student level, teachers must not only have the knowledge of how to evaluate the type of data discussed for Indicator 3.3.1, but also the knowledge of formative assessment practices that can provide insight into student misconceptions and faulty thinking. For example, when an individual student demonstrates continued poor performance and the previously mentioned efforts have not led to success, then the use of mathematics curriculum-based assessment techniques can be utilized to pinpoint reasons for the student's mathematics difficulties. For example, the use of error-pattern analyses, flexible mathematics interviews, and CRA assessments can effectively target what a student is able and unable to do given a mathematics concept or skill and can identify faulty areas of mathematical thinking that impact the student's mathematical progress (e.g., Bryant, 1996; Gersten, 1998; Ginsburg, 1987; Howell et al., 1993; Kamii, 1985; Liedtke, 1988; Mercer & Mercer, 2005; Van de Walle, 2005; Zigmond, Vallecorsa, & Silverman,1981). See Indicator 2.2 for a discussion about error pattern. The flexible mathematics interview/student interview is a valuable tool for gaining insights into students' mathematical thinking. Although there is a variety of approaches a teacher can use to conduct a flexible interview (Van de Walle & Lovin, 2006), the key is engaging students in examining a mathematical situation (e.g., student's math work, a mathematical model, a problem to be solved) and obtaining their thoughts about the mathematical situation, including what they did and why they did what they did, what they think a mathematical model may represent, and why or how they would go about solving a problem. The teacher obtains data about the student's thinking from what the student says and does. Even when students correctly respond to a mathematical task, their thinking may not be accurate. For example, a student may accurately solve 1/2 x 1/4 by writing 1/8 as the solution. However, during a flexible interview, the student may indicate the reason 1/8 is the correct solution is because 2 x 4 (i.e., the denominators) is 8; therefore, that is why 1/8 is the correct answer. By probing a little more using area-model-fraction manipulatives, the teacher may realize that the student knows the algorithm but cannot show why 1/2 x 1/4 is 1/8 (e.g., that 1/2 of a 1/4 piece equals a 1/8 piece) or even that 1/8 is less than 1/4. With some more probing, the teacher may determine that the student does not clearly understand the reciprocal relationship between multiplication and division and how this operates while multiplying fractions using the traditional algorithm. Allsopp, Kyger, Lovin, Gerretson, and Ray (2008) described a process for utilizing data gathered through the integration of CRA assessment, error-pattern analysis, and a flexible mathematics interview to develop an instructional hypothesis to guide interventions by pinpointing what students can do and what they cannot do and why. Fuchs and colleagues (2008) described an assessment process by which a mathematics learning task is given to a student and then instruction or feedback is provided to help the student learn the task. The teacher records the student's response to the instruction/feedback as a way to determine the potential for learning the given skill set. If areas of mathematical difficulties for an individual student are documented via reliable screening and curriculum-based measurement processes, then the use of informal curriculum based assessment techniques can provide instructionally relevant data to guide subsequent instruction and intervention. The value of these types of informal formative assessments is that they allow the teacher to identify misconceptions and errors and rapidly detect and re-teach where misunderstandings occur using the types of explicit systematic mathematics instruction practices discussed for Indicator 3.2.3.

Conclusion

We have attempted to describe in great detail the evidence-based indicators included in this IC so that SEAs, IHEs, and other stakeholders have a substantive description of the characteristics of high-quality mathematics instruction for students with disabilities and other struggling learners. SEAs, IHEs, and other stakeholders may use the IC to evaluate and improve current state licensure requirements; state-, district-, and school-level PD activities; and teacher preparation for pre-K-12 mathematics for the benefit of students with disabilities and other struggling learners. There are strengths and weaknesses to the research base related to effective mathematics instruction. We have attempted to identify and describe the practices we believe are foundational to affecting positive mathematics outcomes for students. Some practices have a stronger evidence base than others, and the level of evidence is indicated in the IC (see Appendix). The indicators and related practices described in this narrative can assist any SEA or IHE to critically evaluate their current practices, set goals for improvement, develop and implement improvement plans, and set benchmarks for evaluating improvement.

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Appendix

Innovation Configuration for Mathematics

Essential Components	**Implementation Levels**				
Instructions: Place an X under the appropriate variation implementation score for each course syllabus that meets the criteria level from 0 to 3. Score and rate each item separately.	**Level 0** There is no evidence that the component is included in the syllabus, or the syllabus only mentions the component.	**Level 1** Must contain at least one of the following: reading, test, lecture/presentation, discussion, modeling/ demonstration, or quiz.	**Level 2** Must contain at least one item from Level 1, plus at least one of the following: observation, project/activity, case study, or lesson plan study.	**Level 3** Must contain at least one item from Level 1 as well as at least one item from Level 2, plus at least one of the following: tutoring, small group student teaching, or whole group internship.	**Rating** Rate each item as the number of the highest variation receiving an X under it.
1.0 Teacher Readiness: Mathematics Content: Core, Supplemental, and Intensive					
1.1 - Demonstrate competency in and understand the underlying concepts for the mathematics content they teach or will be certified to teach. 1.2 - Demonstrate understandings of mathematical concepts and skills within and across domains (e.g., counting and cardinality, operations and algebraic thinking) and how they interrelate and build upon one another over time (e.g., mathematics progressions). 1.3 - Know and can engage in the eight critical practices emphasized by the Common Core State Standards (CCSS) to promote mathematical understanding, reasoning, and problem solving.					

Essential Components	Implementation Levels				
Instructions: Place an X under the appropriate variation implementation score for each course syllabus that meets the criteria level from 0 to 3. Score and rate each item separately.	**Level 0**	**Level 1**	**Level 2**	**Level 3**	**Rating**
	There is no evidence that the component is included in the syllabus, or the syllabus only mentions the component.	Must contain at least one of the following: reading, test, lecture/presentation, discussion, modeling/ demonstration, or quiz.	Must contain at least one item from Level 1, plus at least one of the following: observation, project/activity, case study, or lesson plan study.	Must contain at least one item from Level 1 as well as at least one item from Level 2, plus at least one of the following: tutoring, small group student teaching, or whole group internship.	Rate each item as the number of the highest variation receiving an X under it.
1.0 Teacher Readiness: Mathematics Content: Core, Supplemental, and Intensive					
1.4 - Demonstrate an understanding of effective teaching practices specific to mastery of particular learning goals, content, and student proficiency.					

Essential Components	Implementation Levels				
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2.0 Teacher Readiness: Student Learning: Core, Supplemental, and Intensive					
2.1 - Demonstrates understanding of how typical students' mathematical thinking develops over time for foundational concepts. 2.2 - Demonstrates understandings of common mathematical misconceptions and error patterns that represent faulty mathematical thinking. 2.3 - Demonstrate understandings of potential barriers to learning mathematics for students with disabilities.					

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3.0 Planning Instruction: Deciding What to Teach: Core					
3.1.1 - Specifies the sequence of expected mathematics learning outcomes and places these learning outcomes on an instructional timeline with explicit consideration of multiyear learning goals. 3.1.2 - Utilizes screening assessment to determine whether systemic learning deficits exist (class wide, grade wide, course specific; e.g., Algebra 1). 3.1.3 - Emphasizes critical areas of mathematics foundational to mathematical success (e.g., for grades pre-K-8: number sense, number operations, and algebraic thinking) by targeting several big ideas per grade level/course-specific domains (e.g., Algebra, Geometry) for in-depth emphasis and continuous progress monitoring across the school year. 3.2.1 - Incorporates activities to develop					

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3.0 Planning Instruction: Deciding What to Teach: Core					
conceptual understanding. 3.2.3 - Utilizes explicit systematic instructional practices that support mathematical learning gains for students who require more intensive instruction. 3.3.1 - Engages in routine monitoring of students' mastery of key mathematics concepts and skills at the student, classroom, and grade levels to guide instructional decisions.					

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4.0 Planning Instruction: Deciding What to Teach: Supplemental (in addition to Core)					
4.1 - Utilizes screening assessment data to identify students who need supplemental support. 4.2 - Utilizes assessment of just-taught skills to identify students who require supplemental instruction to master essential skills. 4.3 - Instructional content is aligned with students' levels of understanding and proficiency, which may require working on prerequisite mathematics concepts/skills acquisition and fluency. 4.4 - Collaboratively works with school personnel to identify and appropriately integrate supplemental mathematics instruction within the instructional schedule including utilization of appropriate instructional supports (e.g., support facilitation, accommodations, assistive					

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	There is no evidence that the component is included in the syllabus, or the syllabus only mentions the component.	Must contain at least one of the following: reading, test, lecture/presentation, discussion, modeling/ demonstration, or quiz.	Must contain at least one item from Level 1, plus at least one of the following: observation, project/activity, case study, or lesson plan study.	Must contain at least one item from Level 1 as well as at least one item from Level 2, plus at least one of the following: tutoring, small group student teaching, or whole group internship.	Rate each item as the number of the highest variation receiving an X under it.
4.0 Planning Instruction: Deciding What to Teach: Supplemental (in addition to Core)					
technology).					

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5.0 Planning Instruction: Deciding What to Teach: Intensive (in addition to Core and Supplemental)					
5.1 - Utilizes assessment data that targets foundational mathematics concepts/skills related to the core curriculum to pinpoint gaps in understandings for individual students. 5.2 - Prioritizes target foundational mathematics concepts/skills for instruction (based on sequences of instructional skills) to effectively and efficiently assist students to strengthen their overall mathematics competence. 5.3 - Utilizes assessments that include strategies to verify conceptual understanding (e.g., asking students to think aloud while solving a problem, draw the solution, problem solve using concrete materials). 5.4 - Uses assessments to verify mastery of prerequisite skills and to verify the effect of					

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5.0 Planning Instruction: Deciding What to Teach: Intensive (in addition to Core and Supplemental)					
potential intervention supports (e.g., guided practice, use of incentives, fluency building for prerequisite skills). 5.5 - Collaboratively works with school personnel to identify and appropriately schedule instructional time for intensive instruction including utilization of appropriate instructional supports (e.g., individual intervention, accommodations).					

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6.0 Instructional Planning: Deciding How to Teach: Core and Supplemental					
6.1 - Designs instruction to prevent/re-teach common misconceptions and errors in students' mathematical thinking of grade-level/course-specific content. 6.2 - Designs instruction to facilitate students' development of connections and understandings of relationships among mathematics concepts. 6.3 - Situates the learning of target mathematics concepts/skills within authentic/meaningful contexts relevant to students' interests and daily lives. 6.4 - Incorporates activities to develop conceptual understanding, including identifying and explaining patterns; developing hypotheses or predictions; testing, proving, generalizing, and refuting; and providing mathematical proofs to verify correct responses and to connect					

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6.0 Instructional Planning: Deciding How to Teach: Core and Supplemental					
understandings. 6.5 - Provides sufficient opportunities to build fluency for concepts and skills at grade level and maintain proficiency of prerequisite skills during instruction. 6.6 - Integrates the use of teaching tools and technology appropriately (e.g., when the use of manipulatives and a particular technology are likely to advance mathematical understandings and skills and when they may not) with core instruction to establish understanding. 6.7 - Focuses mathematics instruction to address where students are in terms of acquiring understanding, building proficiency, maintaining proficiency over time, generalizing, and adapting existing mathematical knowledge to make new knowledge.					

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6.0 Instructional Planning: Deciding How to Teach: Core and Supplemental					
6.8 - Flexibly adjusts the nature of teaching and learning supports during instruction to address the diverse learning needs of students (e.g., more teacher directed to more student directed, more explicit to more implicit).					

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7.0 Instructional Planning: Deciding How to Teach: Intensive (in addition to Core and Supplemental)					
7. 1 - Utilizes highly explicit acquisition and proficiency-oriented intervention practices, including extensive modeling of correct responses, explicit instruction for conceptual understanding, immediate and more elaborate and individualized corrective feedback, more gradual task difficulty increases, and more gradual fading of support for correct responding. 7.2 - Systematically utilizes a teacher-directed, explicit, and systematic concrete-representation-abstract sequence of instruction. 7.3 - Incorporates frequent assessment of student understanding during instruction to monitor students' mathematical thinking and adjust instruction accordingly. 7.4 - Explicitly teaches concept/skill-specific mathematics learning strategies as					

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7.0 Instructional Planning: Deciding How to Teach: Intensive (in addition to Core and Supplemental)					
appropriate. 7.5 - Utilizes appropriate instructional games and self-correcting materials to engage students in building proficiency and maintenance of target foundational mathematics concepts/skills at appropriate concrete, representational, and abstract levels of understanding. 7.6 - Integrates brief fluency probes to increase or maintain students' proficiencies and monitor progress with basic prerequisite skills related to the core curriculum that are not targeted for primary intensive instruction (e.g., computations and operations).					

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8.0 Evaluating Instructional Effects and Adjusting Subsequent Instruction: Core					
8.1 - Utilizes periodic assessment to verify retention of learned skills. 8.2 - Utilizes annual assessment for accountability linked to system planning and problem solving. 8.3 - Engages in routine monitoring of student mastery of key mathematics concepts/skills and compares classroom student data to grade level data for all students in that grade for instructional decision making. 8.4 - Utilizes assessment data to select instruction practices aligned with students' levels of conceptual understanding and proficiency of mathematics learning objectives. 8.5 - Effectively collaborates with school-based problem-solving teams to					

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8.0 Evaluating Instructional Effects and Adjusting Subsequent Instruction: Core					
describe nature of core instruction, share informal classroom assessment data, and make appropriate instructional decisions.					

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9.0 Evaluating Instructional Effects and Adjusting Subsequent Instruction: Supplemental (in addition to Core)					
9.1 - Conducts/utilizes follow-up assessment to determine when supplemental intervention has been successful and can be discontinued or when students should be transitioned to more intensive intervention procedures. 9.2 - Effectively collaborates with school-based problem-solving teams to describe nature of supplemental instruction, share performance data, and make appropriate instructional decisions.					

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10.0 Evaluating Instructional Effects and Adjusting Subsequent Instruction: Intensive (in addition to Core and Supplemental)					
10.1 - Conducts/utilizes follow-up assessment conducted to verify gains for targeted foundational mathematics concepts/skills. Conducts/utilizes follow-up assessment data to verify generalized learning improvements during core instruction (e.g., verifying children perform outside of risk range on subsequent screenings and score in the proficient range on year-end accountability tests). 10.2 - Effectively collaborates with school-based problem-solving teams to describe nature of intensive instruction, share performance data, and make appropriate instructional decisions.					